Statistical Inference-II

Testing

Best critical region :-

If $A_1, A_2, A_3, \ldots, A_n$ are the finite number of critical region of size "α" then the critical region having

i) Minimum type II error .

ii) Power of test based on critical region which is maximum is called best critical region or most powerful region. Amongest all the critical region of size "α",one for which 1-β is maximum is called best critical region.

Most powerful test (MPT):-

The test based on best critical region of size "α"is called most powerful test. A good test is that for which

i) $\prod_{i=1}^{n} \theta_0$ =P[Reject $H_0 \big/ H_0$ Is true]=α

ii) $\prod_{i=1}^{n} \theta_1$ =P[Reject $H_0 \big/ H_0$ Is false]=1-β

Neyman Pearson Lemma :-

let $X_1, X_2, X_3, \ldots, X_n$ be a sample from f(x;Θ) then its likelihood function L.H.F is

$$L(\underline{X};\theta) = \prod_{i=1}^{n} f(X;\theta)$$

Suppose θ_0 and θ_1 are two distinct fixed values of 'Θ' , K be a positive constant "C*" is a subset of sample space 'S' such that

$i) \dfrac{L(\underline{X};\theta_0)}{L(\underline{X};\theta_1)} \le K \qquad \forall \underline{X} \in C^*$

ii) $\dfrac{L(\underline{X};\theta_1)}{L(\underline{X};\theta_0)} \ge K \qquad \forall \underline{X} \in C^*$

iii) $P(X \in C^*) = \alpha$

Let θ_0 and θ_1 are two distinct values. A test corresponding to C* is the most powerful test of 'α'.

Procedure :-

Whit the p.d.f of the distribution take its likelihood function .if we are testing $H_0 : \theta = \theta_0$ and H_1 or $H_A : \theta = 1$. Then obtain likelihood function $L(H_0)$ and $L(H_A)$.

The Neymen Pearson Lemma is

$$\frac{L(H_0)}{L(H_1)} \leq K$$

Where 'K' is any constant to test the hypothesis define either of the two critical

Taking likelihood function

$$L(X;\mu\ ) = {e^{-n\mu}\mu^{\sum X}} \Big/ {\prod_{i=1}^{n} X!}$$

Now taking likelihood function at $H_0 : \mu = \mu_0$

$$L(X;\mu_0) = {e^{-n\mu_0}\mu_0^{\sum X}} \Big/ {\prod_{i=1}^{n} X!}$$

And likelihood function at $H_A : \mu = \mu_1$

$$L(X;\mu_1) = {e^{-n\mu_1}\mu_1^{\sum X}} \Big/ {\prod_{i=1}^{n} X!}$$

By Neymen Pearson Lemma theorm

$$\frac{L(H_0)}{L(H_1)} \leq K\alpha$$

$$\frac{e^{-n\mu_0}\mu_0^{\sum X} \Big/ \prod_{i=1}^{n} X!}{e^{-n\mu_1}\mu_1^{\sum X} \Big/ \prod_{i=1}^{n} X!} \leq K\alpha$$

$$\frac{e^{-n\mu_0}\mu_0^{\sum X}}{e^{-n\mu_1}\mu_1^{\sum X}} \leq K\alpha$$

$$(\frac{\mu_0}{\mu_1})^{\sum X} e^{-n\mu_0 + n\mu_1} \leq K\alpha$$

$$(\frac{\mu_0}{\mu_1})^{\sum X} \leq \frac{k\alpha}{e^{-n\mu_0 + n\mu_1}}$$

$$(\frac{\mu_0}{\mu_1})^{\sum X} \leq k\alpha e^{-n\mu_0 + n\mu_1}$$

Taking log on both sides

$$Log(\frac{\mu_0}{\mu_1})^{\sum X} \leq \log[k\alpha e^{-n\mu_0+n\mu_1}]$$

$$\sum X Log(\frac{\mu_0}{\mu_1}) \leq \log k\alpha + \log e^{-n\mu_0+n\mu_1}$$

$$\sum X[Log\mu_0 - \log \mu_1] \leq \log k\alpha + n\mu_0 - n\mu_1$$

$$\therefore \sum X = n\overline{X}$$

If $\theta_1 < \theta_0$ then

$$\overline{X} \leq \frac{1}{n}[\log k\alpha + n\mu_0 - n\mu_1] \Big/ [\log \mu_0 - \log \mu_1]$$

Where

$$C'' \leq \frac{1}{n}[\log k\alpha + n\mu_0 - n\mu_1] \Big/ [\log \mu_0 - \log \mu_1]$$

SO

$$\overline{X} \leq C''$$

If $\theta_1 > \theta_0$ then

$$-\overline{X}[Log\mu_1 - \log \mu_0] \leq \frac{1}{n}[\log k\alpha + n\mu_0 - n\mu_1]$$

$$-\overline{X}[Log\mu_1 - \log \mu_0] \leq \frac{1}{n}[\log k\alpha e^{n\mu_0 - n\mu_1}]$$

Multiply '-1' throughout the equation

$$\overline{X}[Log\mu_1 - \log \mu_0] \geq -\frac{1}{n}[\log k\alpha e^{n\mu_0 - n\mu_1}]$$

$$\overline{X} \geq -\frac{1}{n}[\log k\alpha e^{n\mu_0 - n\mu_1}] \Big/ [Log\mu_1 - \log \mu_0]$$

Where

$$C' \geq -\frac{1}{n}[\log k\alpha e^{n\mu_0 - n\mu_1}] \Big/ [Log\mu_1 - \log \mu_0]$$

SO

$$\overline{X} \geq C'$$

Question 2

let $X_1, X_2, X_3, \ldots, X_n$ be a random sample of size 'n' from a normal distribution with mean zero and variance σ^2. Find the best critical region of size α=0.05 . for testing $H_0 : \sigma^2 = 1$ vs $H_1 : \sigma^2 = 2$. In this best critical region of size α=0.05 . for testing $H_0 : \sigma^2 = 1$ vs $H_1 : \sigma^2 = 4$. Also find power of test .

Solution

AS $X \sim N(0,\sigma^2)$

$$f(X;0,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{X-\mu}{\sigma})^2}$$

$$f(X;0,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{X}{\sigma})^2}$$

Then likelihood function is

$$L(X;0,\sigma^2) = (\frac{1}{\sigma})^n (\frac{1}{\sqrt{2\pi}})^n e^{-\frac{1}{2}\frac{\sum X^2}{\sigma^2}}$$

As $H_0 : \sigma^2 = 1$

$$L(H_0) = (\frac{1}{\sqrt{2\pi}})^n e^{-\frac{1}{2}\sum X^2}$$

As $H_A : \sigma^2 = 2$

$$L(H_A) = (\frac{1}{\sqrt{2}})^n (\frac{1}{\sqrt{2\pi}})^n e^{-\frac{1}{2}\sum X^2 / 2}$$

$$L(H_A) = (\frac{1}{\sqrt{2}})^n (\frac{1}{\sqrt{2\pi}})^n e^{-\sum X^2 / 4}$$

By Neymen Pearson Lemma theorm

$$\frac{L(H_0)}{L(H_1)} \le K\alpha$$

$$\frac{(\frac{1}{\sqrt{2\pi}})^n e^{-\sum X^2 / 2}}{(\frac{1}{\sqrt{2}})^n (\frac{1}{\sqrt{2\pi}})^n e^{-\sum X^2 / 4}} \le K\alpha$$

$$e^{-\sum X^2 / 2 + \sum X^2 / 4} \le K\alpha(\frac{1}{\sqrt{2}})^n$$

$$e^{-\sum X^2 / 4} \le K\alpha(\frac{1}{\sqrt{2}})^n$$

Taking log on both sides

$$-\sum X^2 / 4 \le \log(K\alpha(\frac{1}{\sqrt{2}})^n)$$

$$-\sum X^2 \le 4\log(K\alpha(\frac{1}{\sqrt{2}})^n)$$

Multiply by -1 on both sides

$$\sum X^2 \geq -4\log(K\alpha(\frac{1}{\sqrt{2}})^n)$$

$$\sum X^2 \geq C$$

Is the best critical region to test $H_0:\sigma^2=1$ vs $H_1:\sigma^2=2$.

Probability of type-I error is

P[Reject H_0/H_0 Is true]=α

$$P[\sum X^2 \geq C/H_0] = \alpha$$

$$P[\sum_{i=1}^{10}(X-0)^2/\sigma^2 \geq C/\sigma^2]_{H_0} = \alpha$$

$$P[\chi^2_{(10)} \geq \frac{18.31}{1}] = \alpha$$

$$P[\chi^2_{(10)} \geq 18.31] = \alpha$$

AS $\chi^2_{(10)} \geq 18.31$ Is best critical region for test $H_0:\sigma^2=1$ vs $H_1:\sigma^2=2$ at α=5%.

Power of test :

1-β=P[Reject H_0/H_A Is true]

$$1-\beta = P[\sum X^2 \geq C/H_A]$$

$$= P[\sum_{i=1}^{10}(X-0)^2/\sigma^2 \geq C/\sigma^2]_{H_A}$$

$$= P[\chi^2_{(10)} \geq \frac{18.31}{2}]$$

$$= P[\chi^2_{(10)} \geq 9.155]$$

Again $H_0:\sigma^2=1$ vs $H_1:\sigma^2=4$

As $X \sim N(0,\sigma^2)$

$$f(X;0,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}X^2/\sigma^2}$$

Then likelihood function is

$$L(X;0,\sigma^2) = (\frac{1}{\sigma})^n(\frac{1}{\sqrt{2\pi}})^n e^{-\frac{1}{2}\frac{\sum X^2}{\sigma^2}}$$

As $H_0:\sigma^2=1$

$$L(H_0) = (\frac{1}{\sqrt{2\pi}})^n e^{-\frac{1}{2}\sum X^2}$$

AS

$$H_A : \sigma^2 = 4$$

$$L(H_A) = (\frac{1}{2})^n (\frac{1}{\sqrt{2\pi}})^n e^{-\sum X^2 / 8}$$

By Neymen Pearson Lemma theorm

$$\frac{L(H_0)}{L(H_1)} \leq K\alpha$$

$$\frac{(\frac{1}{\sqrt{2\pi}})^n e^{-\sum X^2 / 2}}{(\frac{1}{2})^n (\frac{1}{\sqrt{2\pi}})^n e^{-\sum X^2 / 8}} \leq K\alpha$$

$$e^{-\sum X^2 / 2 + \sum X^2 / 8} \leq K\alpha(\frac{1}{2})^n$$

$$e^{-3\sum X^2 / 8} \leq K\alpha(\frac{1}{2})^n$$

Taking log on both sides

$$-3\sum X^2 / 8 \leq Log[K\alpha(\frac{1}{2})^n]$$

$$\sum X^2 \geq -\frac{8}{3} Log[K\alpha(\frac{1}{2})^n]$$

$$\sum X^2 \geq C$$

Is the best critical region to test $H_0 : \sigma^2 = 1$ vs $H_1 : \sigma^2 = 4$.

QUESTION

Given the following distribution $f(X;\theta)=\frac{1}{\theta}$ 0≤X≤ϴ and test the hypothes $H_0:\theta=1$ vs $H_A:\theta=2$ .BY mean of single observation that is value of 'X' what would be the size of 'α'. Probability of omitting the type 1st error and what is the probability of omitting type 2nd error β.If we choose the interval

i)X≥0.5

ii)1≤X≤1.5

also obtain power of test.

solution

i)As we know that

α=P[type I error]

$$\alpha = P[X \in C/_{H_0}]$$

$$\alpha = P[X \geq 0.5/_{\theta=1}]$$

$$\alpha = P[0.5 \leq X \leq 1/\theta=1]$$

$$\alpha = \int_{0.5}^{1} f(x)\,dx/\theta=1]$$

$$\alpha = \int_{0.5}^{1} \frac{1}{\theta}\,dx/\theta=1]$$

$$\alpha = \int_{0.5}^{1} 1.dx$$

$$\alpha = x\Big|_{0.5}^{1}$$

$$\alpha = 0.5$$

β=P[type II error]

$$= P[X \in C^*/_{H_A}]$$

$$= P[X \leq 0.5/_{H_A}]$$

$$= P[X \leq 0.5/_{\theta=2}]$$

$$= P[0 \leq X \leq 0.5/\theta=2]$$

$$= \int_0^{0.5} f(x)dx / \theta = 2]$$

$$= \int_0^{0.5} \frac{1}{\theta} dx / \theta = 2]$$

$$= \frac{1}{2} \int_0^{0.5} 1.dx$$

$$= \frac{1}{2} [0.5 - 0]$$

$$\beta = 0.25$$

Power of test :

1-β= 1-0.25

= 0.75

ii) 1≤x≤1.5

α=P[type I error]

$$\alpha = P[X \in C / H_0]$$

$$\alpha = P[1 \le X \le 0.5 / \theta = 1]$$

$$\alpha = \int_{0.5}^{1} f(x)\, dx / \theta = 1]$$

$$\alpha = \int_1^{1.5} \frac{1}{\theta} dx / \theta = 1]$$

At Θ=1

$$\alpha = 0$$

β=P[type II error]

$$= P[X \in C^* / H_A]$$

$$= 1 - P[1 \le X \le 1.5 / \theta = 2]$$

$$= 1 - \int_1^{1.5} f(x)dx / \theta = 2]$$ at Θ=2

$$= 1 - \int_1^{1.5} \frac{1}{\theta} dx]$$

$$= 1 - \frac{1}{2} \int_1^{1.5} 1.dx$$

$$=1-\frac{1}{2}x\Big|_{1}^{1.5}$$

$$=1-\frac{1}{2}[1.5-1]$$

$$=1-\frac{1}{2}[0.5]$$

$$=1-0.25$$

$$\beta=0.75$$

Now

$$1-\beta=1-0.75$$

$$1-\beta=0.25$$

Question

Let 'P' be the probability that a coin will turn up head in a single time. In order to test Ho: P=1/2

Vs Ho: P=3/4, coin is tossed 5 times and Ho is reject it means that more than 3 head $X \geq 4$. Find

The probability of type 1st and 2nd error. Also find "$1-\beta$".

solution

Let X denotes the number of heads in testing a coin.

The given p.d.f is

$$P(X = x) = (C_X^5)P^X(1-P)^{5-X} \qquad \rightarrow (A) \qquad \therefore n = 5$$

The critical region is

$$C = \{x; x \geq 4\} \quad and \quad C^* = \{x; x \leq 3\}$$

Now

α = [Type 1st error]

$$\alpha = P[x\varepsilon C / Ho]$$

=P [reject Ho/Ho is true]

= P[$x \geq 4$] at P=1/2

=P[x=4 at P=1/2] + P[x=5 at P=1/2]

$$= (C_4^5)(1/2)^4(1/2) + (C_5^5)(1/2)^5$$

= (5/32) + (1/32) = 6/32

α = 3/16

$$\beta = P[x \in C^* / H_A]$$

β = [Type 2nd error]

=P[$x < 4$]

= 1-P$[x \geq 4]_{P=3/4}$

$$1 - [(C_4^5)(3/4)^4(1/4) + (C_5^5)(3/4)^5]$$

$$1 - \left[\frac{405}{1024} + \frac{243}{1024}\right]$$

$$1 - 0.6328$$

$$0.3672$$

Power of test:

$$1 - \beta = 1 - 0.3672$$

$$= 0.6328$$

Question :5

let $X_1, X_2, X_3, \ldots, X_n$ be a random sample from a density $f(X;\theta)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(X-\theta)^2}$. Find the best critical region and test the hypothesis $H_0:\theta=0$ vs $H_1:\theta=1$.

Solution

The given p.d.f is

$$f(X;\theta)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(X-\theta)^2}$$

Taking L.H.F

$$L(X;\theta)=(\frac{1}{\sqrt{2\pi}})^n e^{-\frac{1}{2}\sum(X-\theta)^2}$$

As $H_0:\theta=0$

$L(H_0)=(\frac{1}{\sqrt{2\pi}})^n e^{-\frac{1}{2}\sum X^2}$ As $H_A:\theta=1$ $L(H_1)=(\frac{1}{\sqrt{2\pi}})^n e^{-\frac{1}{2}\sum(X-1)^2}$

By Neymen Pearson Lemma theorm

$$\frac{L(H_0)}{L(H_1)}\le K\alpha$$

$$\frac{(\frac{1}{\sqrt{2\pi}})^n e^{-\sum X^2/2}}{(\frac{1}{\sqrt{2\pi}})^n e^{-\frac{1}{2}\sum(X-1)^2}}\le K\alpha$$

$e^{-\sum X^2/2+\frac{1}{2}\sum(X-1)^2}$ Taking log on both sides

$$-\frac{\sum X^2}{2}+\frac{\sum X^2}{2}+\frac{n}{2}-\sum X\le LogK\alpha$$

$$-\sum X\le LogK\alpha-\frac{n}{2}$$

$$-n\overline{X}\le LogK\alpha-\frac{n}{2}$$

Multiplying by '-1'0n both sides

$$n\overline{X}\ge\frac{n}{2}(-LogK\alpha)$$

$\overline{X}\ge\frac{1}{n}[\frac{n}{2}-LogK\alpha]$ $\therefore C=\frac{1}{n}[\frac{n}{2}-LogK\alpha]$

$$\overline{X}\ge C$$

Is the best critical region to test $H_0:\theta=0$ vs $H_1:\theta=1$.

Question 6:

Let $x_1, x_2, x_3, \ldots., x_n$ be a random sample from the density $f(x,\theta) = \theta e^{\theta x}$ $0 \le x \le \infty$. find the best critical region. Test the hypothesis $H_0 : \theta = \theta_0 \quad vs \quad H_A : \theta = \theta_1$.

Solution:

The given p.d.f.

$$f(x,\theta) = \theta e^{\theta x}$$

Taking likelihood function

$$L(x) = \theta^n e^{-\theta \Sigma x}$$

As $H_0 : \theta = \theta_0$

$$L(H_0) = \theta_0{}^n e^{-\theta_0 \Sigma x}$$

And $H_A : \theta = \theta_1$ $\qquad L(H_A) = \theta_1{}^n e^{-\theta_1 \Sigma x}$

By Nyman Pearson lemma theorem:

$$\frac{L(H_0)}{L(H_A)} \le k_\alpha$$

$$\frac{\theta_0{}^n e^{-\theta_0 \Sigma x}}{\theta_1{}^n e^{-\theta_1 \Sigma x}} \le k_\alpha$$

$$\left(\frac{\theta_0}{\theta_1}\right)^n e^{-\theta_0 \Sigma x + \theta_1 \Sigma x} \le k_\alpha$$

$$e^{-\theta_0 \Sigma x + \theta_1 \Sigma x} \le \left(\frac{\theta_0}{\theta_1}\right)^n k_\alpha$$ Taking log on both sides:

$$-\theta_0 \Sigma x + \theta_1 \Sigma x \le \log\left[\left(\frac{\theta_0}{\theta_1}\right)^n k_\alpha\right]$$

$$\Sigma x(\theta_1 - \theta_0) \le \log\left[\left(\frac{\theta_0}{\theta_1}\right)^n k_\alpha\right]$$

$$\Sigma x \le \log\left[\left(\frac{\theta_0}{\theta_1}\right)^n k_\alpha\right] \Big/ (\theta_1 - \theta_0)$$

Where c= $\log\left[\left(\frac{\theta_0}{\theta_1}\right)^n k_\alpha\right] \Big/ (\theta_1 - \theta_0)$

$$\Sigma x \le c$$

Is the best critical region (BCR) to test $H_0 : \theta = \theta_0 \quad vs \quad H_A : \theta = \theta_1$ when $\theta_1 > \theta_0$.

Question 7:

Let $X_1, X_2, ..., X_n$ be a random sample from the density $f(x;\theta) = \theta^x (1-\theta)^{1-x}$ test the hypothesis $H_o : \theta = \theta_o$ vs. $H_A : \theta = \theta_1$

Solution:

The given p.d.f is

$$f(x;\theta) = \theta^x (1-\theta)^{1-x}$$

Taking likelihood function

$$L(\underline{X}) = \theta^{\Sigma x} (1-\theta)^{n-\Sigma x}$$

As $H_o : \theta = \theta_o$

$$L(H_o) = \theta_o^{\Sigma x} (1-\theta_o)^{n-\Sigma x}$$

As $H_A : \theta = \theta_1$

$$L(H_A) = \theta_1^{\Sigma x} (1-\theta_1)^{n-\Sigma x}$$

By Neymen person lemma theorem

$$\frac{L(H_o)}{L(H_A)} \le K_a$$

$$\frac{\theta_o^{\Sigma x} (1-\theta_o)^{n-\Sigma x}}{\theta_1^{\Sigma x} (1-\theta_1)^{n-\Sigma x}} \le K_a$$

$$\left(\frac{\theta_o}{\theta_1}\right)^{\Sigma x} \left(\frac{(1-\theta_o)}{(1-\theta_1)}\right)^{n-\Sigma x} \le K_a$$

Taking log on both sides

$$\Sigma x \log(\frac{\theta_o}{\theta_1}) + (n-\Sigma x)\log\left(\frac{(1-\theta_o)}{(1-\theta_1)}\right) \le \log K_a$$

$$\Sigma x \log(\frac{\theta_o}{\theta_1}) + n\log\left(\frac{(1-\theta_o)}{(1-\theta_1)}\right) - \Sigma x \log\left(\frac{(1-\theta_o)}{(1-\theta_1)}\right) \le \log K_a$$

$$\Sigma x \log(\frac{\theta_o}{\theta_1}) - \Sigma x \log\left(\frac{(1-\theta_o)}{(1-\theta_1)}\right) \le \log K_a - n\log\left(\frac{(1-\theta_o)}{(1-\theta_1)}\right)$$

$$\Sigma x\left(\log(\frac{\theta_o}{\theta_1}) - \log\left(\frac{(1-\theta_o)}{(1-\theta_1)}\right)\right) \le \log K_a - n\log\left(\frac{(1-\theta_o)}{(1-\theta_1)}\right)$$

$$\Sigma x \le \left. \log K_a - n\log\left(\frac{(1-\theta_o)}{(1-\theta_1)}\right) \middle/ \left(\log(\frac{\theta_o}{\theta_1}) - \log\left(\frac{(1-\theta_o)}{(1-\theta_1)}\right)\right) \right.$$

$$\Sigma x \le C$$

Is the best critical region to test $H_o : \theta = \theta_o$ vs $H_A : \theta = \theta_1$

Question 8:

Let $X_1, X_2, ..., X_n$ be a random sample from the normal distribution with mean 'θ' and variance '100'. Show that $c = \left[x : c \le \overline{X}\right]$ is the best critical region for testing $H_o : \theta = 75$ vs. $H_A : \theta = 78$.Find n and c such that

(i) $p\left[X_1, X_2, ..., X_n \varepsilon \, {}^{c}\!/_{H_o}\right] = p\left[\overline{X} \ge {}^{c}\!/_{H_o}\right] = 0.05$

(ii) $p\left[X_1, X_2, ..., X_n \varepsilon \, {}^{c}\!/_{H_A}\right] = p\left[\overline{X} \ge {}^{c}\!/_{H_A}\right] = 0.90$

As $MISSING$

$$f(X) = \frac{1}{10\sqrt{2\pi}} e^{-\frac{1}{200}(x-\theta)^2}$$

Taking likelihood function

$$L(X) = \left(\frac{1}{10}\right)^n \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{200}\sum(x-\theta)^2}$$

As $H_o : \theta = 75$ let $H_o : \theta = \theta'$

$$L(H_o) = \left(\frac{1}{10}\right)^n \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{200}\sum(x-\theta')^2}$$

As $H_A : \theta = 78$ let $H_1 : \theta = \theta''$

$$L(H_A) = \left(\frac{1}{10}\right)^n \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{200}\sum(x-\theta'')^2}$$

By Neymen Pearson Lemma theorm

$$\frac{L(H_0)}{L(H_1)} \le K\alpha$$

$$e^{-\frac{1}{200}\sum(x-\theta')^2} + e^{-\frac{1}{200}\sum(x-\theta'')^2} \le K_a$$

$$e^{-\frac{1}{200}\sum(x-\theta')^2 + \frac{1}{200}\sum(x-\theta'')^2} \le K_a$$

Taking log on both sides

$$-\frac{1}{200}\Sigma^{(x-\theta')^2} + \frac{1}{200}\Sigma^{(x-\theta'')^2} \le \log K_a$$

$$-\frac{1}{200}\left[\Sigma^{(x-\theta')^2} + \Sigma^{(x-\theta'')^2}\right] \le \log K_a$$

$$-\frac{1}{200}\left[\Sigma x^2 + n\theta'^2 - 2\sum x\theta' - \Sigma x^2 - n\theta'' + 2\theta''\sum x\right] \le \log K_a$$

$$-\frac{1}{200}\left[n\theta'^2 - 2\sum x\theta' - n\theta'' + 2\theta''\sum x\right] \le \log K_a$$

$$\left[-\frac{n\theta'^2}{200}+\frac{2n\overline{X}\theta'}{200}+\frac{n\theta''}{200}-\frac{2\theta''n\overline{X}}{200}\right]\leq \log K_a$$

$$-\frac{2n\overline{X}}{200}[\theta''-\theta']\leq \log K_a+\frac{n\theta'^2}{200}-\frac{n\theta''^2}{200}$$

$$\overline{X}\leq \frac{100\left[\log K_a+\frac{n\theta'^2}{200}-\frac{n\theta''^2}{200}\right]}{n[\theta''-\theta']}$$

Multiplying by -1 so the inequality sign will be changed

$$\overline{X}\geq -\frac{100\left[\log K_a+\frac{n\theta'^2}{200}-\frac{n\theta''^2}{200}\right]}{n[\theta''-\theta']}$$

$\overline{X}\geq$ c

Is the best critical region to test $H_o:\theta=\theta$

$$p\left[X_1,X_2,\ldots,X_n\varepsilon\ c/H_o\right]=p\left[\overline{X}\geq c/H_o\right]=0.05$$

It is given that

$$p\left[\overline{X}\geq c/H_o\right]=0.05\text{ss}$$

$$p\left[\frac{\overline{X}-\theta}{\sigma/\sqrt{n}}\geq \frac{c-\theta}{\sigma/\sqrt{n}}\Big/H_o\right]=0.05$$

$$p\left[Z\geq \frac{\sqrt{n}(c-\theta)}{\sigma}\Big/H_o\right]=0.05$$

$$p\left[Z\geq \frac{\sqrt{n}(c-75)}{10}\Big/H_o\right]=0.05$$

Using area table

$$p[\overline{X}\geq 1.645]=0.05$$

$$\left[\frac{\sqrt{n}(c-75)}{10}\right]=1.645$$

$$\sqrt{n}(c-75)=16.45\rightarrow(1)$$

$$p\left[X_1,X_2,\ldots,X_n\varepsilon\ c/H_A\right]=p\left[\overline{X}\geq c/H_A\right]=0.90$$

$$p\left[\overline{X} \geq c/H_A\right] = 0.90$$

$$p\left[\frac{\overline{X}-\theta}{\sigma/\sqrt{n}} \geq \frac{c-\theta}{\sigma/\sqrt{n}} \Big/ H_o\right] = 0.90$$

$$p\left[Z \geq \frac{\sqrt{n}(c-\theta)}{\sigma}\right] = 0.90$$

$$p\left[Z \geq \frac{\sqrt{n}(c-78)}{\sigma}\right] = 0.90$$

Using inverse area table

$$p[Z \geq -1.28] = 0.90$$

$$\frac{\sqrt{n}(c-78)}{10} = -1.28$$

$$\sqrt{n}(c-78) = -12.8 \rightarrow (2)$$

Subtract eq (1) and (2)

$$\sqrt{n}c - 75\sqrt{n} = 16.45$$

$$\sqrt{n}c - 78\sqrt{n} = -12.8$$

−

$3\sqrt{n}$ =29.25

$$\sqrt{n} = 29.25/3$$

$\sqrt{n}$ =9.75

Square on both sides

N=95.06

Put the value in eq (1)

9.75(c-75)=16.45

9.75c-731.25=16.45

9.75c=747.7

C=747.7/9.75

C=76.6871

Question 9:

Consider the Cauchy distribution f(x)= $\frac{1}{\pi[1+(x-\theta)^2]}$ $\quad -\infty \leq x \leq \infty$

Find BCR for testing $H_0 : \theta = 0 \ \ vs \ \ H_A : \theta = 1$ with the value $k_\alpha = 1$ and n=1.

Solution:

The given p.d.f is

$$f(x) = \frac{1}{\pi[1+(x-\theta)^2]}$$

Taking likelihood function

$$L(x) = \left(\frac{1}{\pi}\right)^n \frac{1}{\prod_{i=1}^{n}[1+(x-\theta)^2]}$$

As $H_0 : \theta = 0$

$$L(H_0) = \left(\frac{1}{\pi}\right)^n \frac{1}{\prod_{i=1}^{n}[1+x)^2]}$$

As $H_A : \theta = 1$

$$L(H_A) = \left(\frac{1}{\pi}\right)^n \frac{1}{\prod_{i=1}^{n}[1+(x-1)^2]}$$

By Nyman Pearson Lemma Theorem:

$$\frac{L(H_0)}{L(H_A)} \leq k_\alpha$$

$$\frac{\frac{1}{\prod_{i=1}^{n}[1+x)^2]}}{\frac{1}{\prod_{i=1}^{n}[1+(x-1))^2]}} \leq k_\alpha$$

$$\frac{\prod_{i=1}^{n}[1+(x-1))^2]}{\prod_{i=1}^{n}[1+x)^2} \leq k_\alpha$$

$$\frac{[1+(x-1)]^2}{[1+x]^2} \leq k_\alpha$$

$$[1+x^2+1-2x \leq 1+x^2$$

$$-2x \leq 1-2$$

$$-2x \leq -1$$

Multiply both sides by -1 and reverse inequality

$$x \leq 1/2$$

Question 10:

Let $x_{1,}x_{2,}x_{3,}....,x_{n,}$ be a random sample of p.d.f. by form $f(x;\theta)=\theta x^{\theta-1}\ 0\le x\le 1$. show that the BCR for testing $H_0\!:\theta=1\ vs\ \ H_A\!:\theta=2$ is of form $c=[\underset{-}{x};c\le\prod_{i=1}^{n}x_i]$.

Solution:

as we know that

$$f(x;\theta)=\theta x^{\theta-1}$$

Taking likelihood function

$$L(x)=\theta^n\prod_{i=1}^{n}x^{\theta-1}$$

$$As\ H_0\!:\theta=1$$

$$L(H_0)=(1)^n\prod_{i=1}^{n}(x)^{1-1}$$

$$=1$$

$$As\ H_A\!:\theta=2$$

$$L(H_A)=(2)^n\prod_{i=1}^{n}(x)^{2-1}$$

$$L(H_A)=2^n\prod_{i=1}^{n}x$$

By Nyman Pearson Lemma Theorem:

$$\frac{L(H_0)}{L(H_A)}\le k_\alpha$$

$$\frac{2^n\prod_{i=1}^{n}x}{1}\le k_\alpha$$

$$\prod_{i=1}^{n}x\le\frac{1}{2^n}k_\alpha \qquad\qquad \therefore c=\frac{1}{2^n}k_\alpha$$

$$\prod_{i=1}^{n}x\le c$$

This is required result.

Question:

Describe Nyman Pearson Lemma

OR

Write the importance of Nyman Pearson Lemma?

A method for constructing best critical region (BCR) is available via a theorem name the Nyman Pearson Lemma.

Let $x_1, x_2, x_3, \ldots\ldots, x_n$ be a random sample from $f(x;\theta)$ where $'\theta'$ is one of the

Two values which is known, $'\theta_0'$ or $'\theta_1'$.such that $[(\theta = \theta_0)\ or\ (\theta = \theta_1)]$ and let $''0 < \alpha < 1''$ be fixed. Let $'k_\alpha'$ be a positive constant and $'c_\alpha'$ be a subset of $'\lambda'$(sample space of the observation) $\theta \in$ which satisfy.

i. $P[(x_1, x_2, x_3, \ldots\ldots, x_n) \in c_\alpha ; \theta_0] = p_{\theta_0}(\underline{x} \in c_\alpha) = \alpha$

While $\alpha = \left[p[\underline{x}\, \varepsilon\, {}^{c_\alpha}\!/\!_{H_0} \right]$

ii. $$\lambda = \frac{p(\underline{x};\theta_1)}{p(\underline{x};\theta_0)} = \frac{L(x_1, x_2, x_3, \ldots\ldots, x_n;\theta_1)}{L(x_1, x_2, x_3, \ldots\ldots, x_n;\theta_0)}$$

$\lambda \geq k_\alpha\ if\ (x_1, x_2, x_3, \ldots\ldots, x_n) \in c_\alpha$

And

$\lambda \leq k_\alpha\ if\ (x_1, x_2, x_3, \ldots\ldots, x_n) \notin c_\alpha$

Then c_α is the best critical region (BCR) of size α for

$H_0 : \theta = \theta_0$

Vs

$H_1 : \theta = \theta_1$